

Nucleon radii and form factors at $Q^2 = 0$ using momentum derivatives

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- Summary

Introduction

- **Proton radius puzzle:** 7σ discrepancy between Muonic hydrogen Lamb shift ($r_E^p = 0.84087(39)$ fm) and atomic hydrogen and scattering experiments using electrons ($r_E^p = 0.8775(51)$ fm)
- Model dependent fit
- $Q_{min}^2 < 0.005$ GeV² from scattering experiment (e.g. from Mainz, Phys.Rev.Lett. 105 (2010) 242001) whereas on the lattice $Q_{min}^2 = 0.05$ GeV² for $V = (5.8 \text{ fm})^3$
- Controversy about finding the radius from fitting scattering data.
 - **"Consistency of electron scattering data with a small proton radius"**, Phys.Lett. B737 (2014) 57-59, Phys.Rev. C93 (2016) no.6, 065207, Phys.Rev. C93 (2016) no.5, 055207
 - **"Solution of the proton radius puzzle? Low momentum transfer electron scattering data are not enough"**, arXiv:1511.00479, Phys.Rev. D92 (2015) no.1, 013013, arXiv:1606.02159
- This motivates the need for a direct calculation of the radius without fitting to form factors.

Rome method: Momentum derivatives of quark propagator

(Phys. Lett. B 718, 589 (2012) [arXiv:1208.5914])

- On a lattice with finite size and quarks satisfying PBC,

$$C_2(\vec{p}, t) = \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \text{Tr} [\langle N(\vec{x}, t) \bar{N}(0) \rangle \Gamma_{pol}] = \sum_{\vec{x}} \text{Tr} \left[\epsilon^{abc} \epsilon^{def} f_{\alpha\gamma\delta\epsilon} f_{\beta\zeta\eta\theta} \right. \\ \left. \left\langle \underbrace{e^{-i\vec{p}\vec{x}} G_{\epsilon\zeta}^{cd}(x, 0)}_{G_{\epsilon\zeta}^{cd}(x, 0; \vec{p})} (G_{\gamma\theta}^{af}(x, 0) G_{\delta\eta}^{be}(x, 0) - G_{\gamma\eta}^{ae}(x, 0) G_{\delta\theta}^{bf}(x, 0)) \right\rangle (\Gamma_{pol})_{\alpha\beta} \right]$$

- $G(x, 0; \vec{p})$ can be obtained by solving the lattice Dirac equation with link variables rescaled by a phase factor,

$$U_k(x) \rightarrow e^{ip_k} U_k(x), \quad \sum_y D(x, y; \vec{p}) G(y, z; \vec{p}) = \delta_{x,z}$$

This is equivalent to a lattice propagator associated with quark field satisfying twisted boundary conditions.

- Allow \vec{p} to be continuous and expand the Dirac operator,

$$D[U, \vec{p}] = D[U] + p_k \frac{\partial D}{\partial p_k} \Big|_{\vec{p}=0} + \frac{p_k^2}{2} \frac{\partial^2 D}{\partial p_k^2} \Big|_{\vec{p}=0} + \dots$$

- Using $DG = 1$ and for clover-improved Wilson action, we find

$$\frac{\partial}{\partial p_k} G(x, y; \vec{p}) \Big|_{\vec{p}=0} = -i \sum_z G(x, z) \Gamma_V^k G(z, y)$$

$$\frac{\partial^2}{\partial p_k^2} G(x, y; \vec{p}) \Big|_{\vec{p}=0} = -2 \sum_{z, z'} G(x, z) \Gamma_V^k G(z, z') \Gamma_V^k G(z', y) -$$

$$\sum G(x, z) \Gamma_T^k G(z, y)$$

$$\Gamma_{V/T}^k G(z, y) \equiv U_k^\dagger(x - \hat{k}) \frac{1 + \gamma^k}{2} G(z - \hat{k}, y) \mp U_k(z) \frac{1 - \gamma^k}{2} G(z + \hat{k}, y)$$

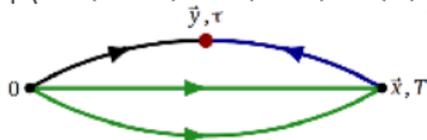
$$\xrightarrow{p_k} = \longrightarrow - ip_k \xrightarrow{\bullet} + (ip_k)^2 \left\{ \xrightarrow{\bullet \bullet} + \frac{1}{2} \xrightarrow{\blacksquare} \right\} + \dots$$

- Extend this to the case of smeared propagators

Momentum derivatives of three-point function

- Three-point function:

$$C_3^\Gamma(\vec{p}, \vec{p}', T, \tau) = \sum_{\vec{x}, \vec{y}} e^{-i\vec{p}'(\vec{x}-\vec{y})} e^{-i\vec{p}\vec{y}} \text{Tr} \left[\left\langle N(\vec{x}, T) O_\Gamma(\vec{y}, \tau) \bar{N}(0) \right\rangle \Gamma_{pol} \right]$$



- Lattice 2016: Momentum derivatives with respect to \vec{p} ($\vec{p}' = 0$)

$$C_3^\Gamma(\vec{p}, T, \tau) \sim \sum_{\vec{y}} \text{Tr} [\langle G_S(\vec{y}) \Gamma G(\vec{y}, 0; \vec{p}) \rangle]$$

$G_S(\vec{y})$: The sequential back-ward propagator (independent of \vec{p}).

- No additional sequential propagators are needed.
- This approach yields noisy second derivatives which motivates the following alternative implementation.

Momentum derivatives of three-point function

- New method: Momentum derivatives with respect to \vec{p}' and \vec{p}

$$\begin{aligned}
 C_3^\Gamma(\vec{p}', \vec{p}, T, \tau) &= \sum_{\vec{x}, \vec{y}} e^{-i\vec{p}'(\vec{x}-\vec{y})} e^{-i\vec{p}\vec{y}} \left\langle \text{Tr}[S_{\Gamma_{pol}}(0; x)G(x, y)\Gamma G(y, 0)] \right\rangle \\
 &= \sum_{\vec{x}, \vec{y}} \left\langle \text{Tr} \left[\left(\gamma_5 G(y, x; \vec{p}') \gamma_5 S_{\Gamma_{pol}}^\dagger(0; x) \right)^\dagger \Gamma G(y, 0; \vec{p}) \right] \right\rangle
 \end{aligned}$$

$S_{\Gamma_{pol}}(0; x)$: sequential source

- We need the following mixed derivatives

$$\begin{aligned}
 &\frac{\partial}{\partial p'^j} \frac{\partial}{\partial p^k} C_3^\Gamma(\vec{p}', \vec{p}, T, \tau) \Big|_{\vec{p}=\vec{p}'=0} = \\
 &\sum_{\vec{x}, \vec{y}} \left\langle \text{Tr} \left[\left(\gamma_5 \frac{\partial}{\partial p'^j} G(y, x; \vec{p}') \Big|_{\vec{p}'=0} \gamma_5 S_{q, \Gamma_{pol}}^\dagger(0; x) \right)^\dagger \Gamma \frac{\partial}{\partial p^k} G(y, 0; \vec{p}) \Big|_{\vec{p}=0} \right] \right\rangle
 \end{aligned}$$

- Additional sequential propagators are needed for every T .

Ground-state contributions :

$$C_2(\vec{p}, t) = e^{-Et} \text{Tr} \left[\langle \Omega | N | p \rangle \langle p | \bar{N} | \Omega \rangle \Gamma_{pol} \right]$$

$$C_3^\mu(\vec{p}', \vec{p}, T, \tau) = e^{-E'(T-\tau)} e^{-E(\vec{p})\tau} \text{Tr} \left[\langle \Omega | N | p' \rangle \langle p' | \bar{q} \gamma^\mu q | p \rangle \langle p | \bar{N} | \Omega \rangle \Gamma_{pol} \right]$$

$$\langle p' | V_q^\mu | p \rangle = \bar{u}(p') \underbrace{\mathcal{F}(\Gamma, \vec{p}', \vec{p})}_{F_1^q \gamma^\mu + F_2^q \frac{i\sigma^{\mu\nu} q_\nu}{2m}} u(p), \quad \text{and} \quad \langle \Omega | N | p \rangle = Z(\vec{p}) [\Gamma u(p)]$$

$$R^\mu(\vec{p}', \vec{p}) = \frac{C_3^\mu(\vec{p}', \vec{p}, T, \tau)}{\sqrt{C_2(\vec{p}', T) C_2(\vec{p}, T)}} \sqrt{\frac{C_2(\vec{p}, T - \tau) C_2(\vec{p}', \tau)}{C_2(\vec{p}', T - \tau) C_2(\vec{p}, \tau)}}$$

$$= \frac{\text{Tr} \left[(m + \not{p}') \mathcal{F}(\gamma^\mu, \vec{p}', \vec{p}) (m + \not{p}) \Gamma_{pol} \right]}{16\sqrt{EE'(E+m)(E'+m)}}$$

The derivative method (1)

Lattice 2016: compute first- and second-order momentum derivatives with respect to the initial state

- Compute $\left. \frac{\partial R^\mu}{\partial p_i} \right|_{\vec{p}=0}$ and $\left. \frac{\partial^2 R^\mu}{\partial p_i^2} \right|_{\vec{p}=0}$ for $\mu = 0, 1, 2$ vector components in x, y and z directions
- For $\Gamma_{\text{pol}} = (1 + \gamma^3 \gamma_5) \frac{1 + \gamma^0}{2}$:

$$\kappa = -2 m \operatorname{Im}(R^2)' - R^0$$

$$r_1^2 = \frac{12 m \operatorname{Im}(R^2)' + 3 R^0 - 12 m^2 (R^0)''}{4 m^2 R^0}$$

Where $r_1^2 = \frac{-6}{F_1} \left. \frac{dF_1}{dQ^2} \right|_{Q^2=0}$

Average over equivalent vector components and directions:

$$(R^2)' = \frac{1}{2}(\partial_1 R^2 - \partial_2 R^1), \quad (R^0)'' = \frac{1}{3}(\partial_1^2 R^0 + \partial_2^2 R^0 + \partial_3^2 R^0)$$

The derivative method (2)

- New method: simultaneously compute first-order momentum derivatives with respect to both initial and final state
- Compute:

$$\frac{\frac{\partial}{\partial p'^j} \frac{\partial}{\partial p^j} C_3^0(\vec{p}', \vec{p}, T, \tau)|_{\vec{p}=\vec{p}'=0}}{C_2(\vec{0}, \tau)} = \frac{1}{4m^2} [F_1 + 2F_2] + \frac{1}{3} F_1 [r_1]^2$$

Removal of excited states

- **Ratio-plateau method:** compute ratio

$$R(T, \tau) = c_{00} + c_{10}e^{-\Delta E_{10}(\vec{p})\tau} + c_{01}e^{-\Delta E_{10}(\vec{p}')(\tau-\tau)}$$

where c_{00} is the desired ground-state matrix element. Then average a fixed number of points around $\tau = T/2$.

- Lattice 2016: $R, \frac{\partial}{\partial p^i} R|_{\vec{p}=0} \sim e^{-\Delta E_{10} T/2}, \frac{\partial^2}{\partial p^{i^2}} R|_{\vec{p}=0} \sim T e^{-\Delta E_{10} T/2}$
- New method: $\frac{\partial}{\partial p'^j} \frac{\partial}{\partial p^k} R|_{\vec{p}=0} \sim e^{-\Delta E_{10} T/2}$

- **Summation method:** compute sums

$$S(T) = \sum_{\tau} R(T, \tau) = b + c_{00}T + dT e^{-\Delta E T} + \dots$$

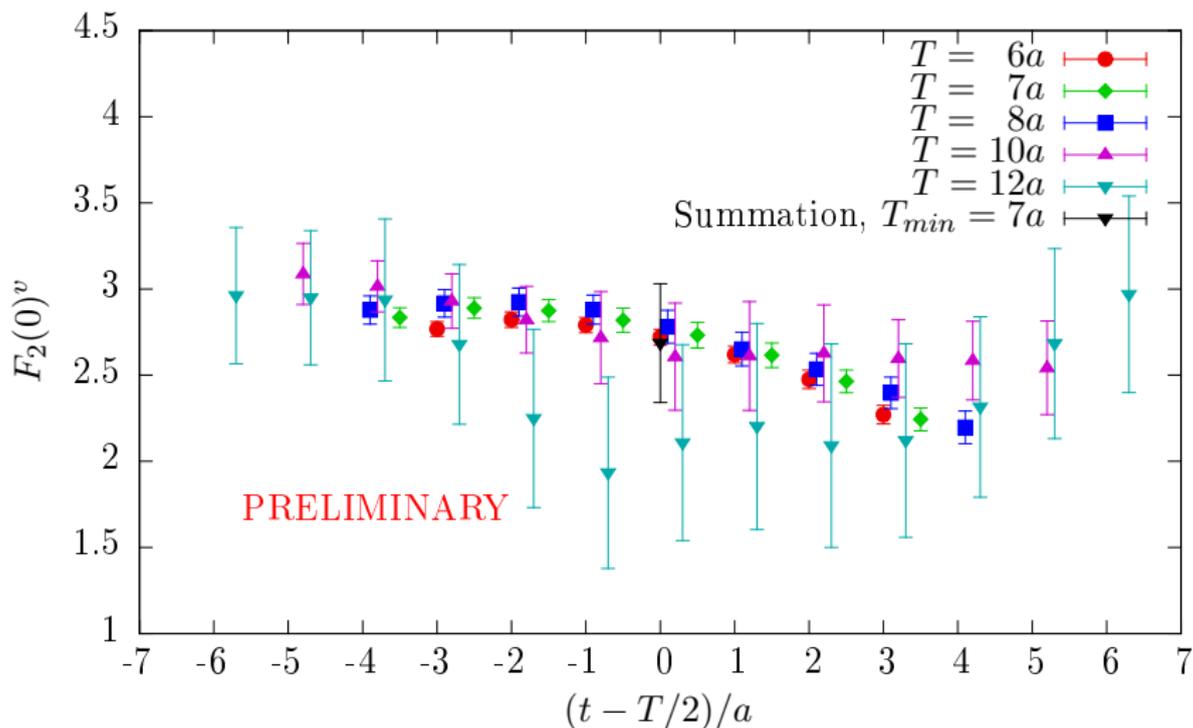
then find their slope, which gives c_{00} .

- Lattice 2016: $S, \frac{\partial}{\partial p^i} S|_{\vec{p}=0} \sim T e^{-\Delta E_{10} T}, \frac{\partial^2}{\partial p^{i^2}} S|_{\vec{p}=0} \sim T^2 e^{-\Delta E_{10} T}$
- New method: $\frac{\partial}{\partial p'^j} \frac{\partial}{\partial p^k} S|_{\vec{p}=\vec{p}'=0} \sim T e^{-\Delta E_{10} T}$

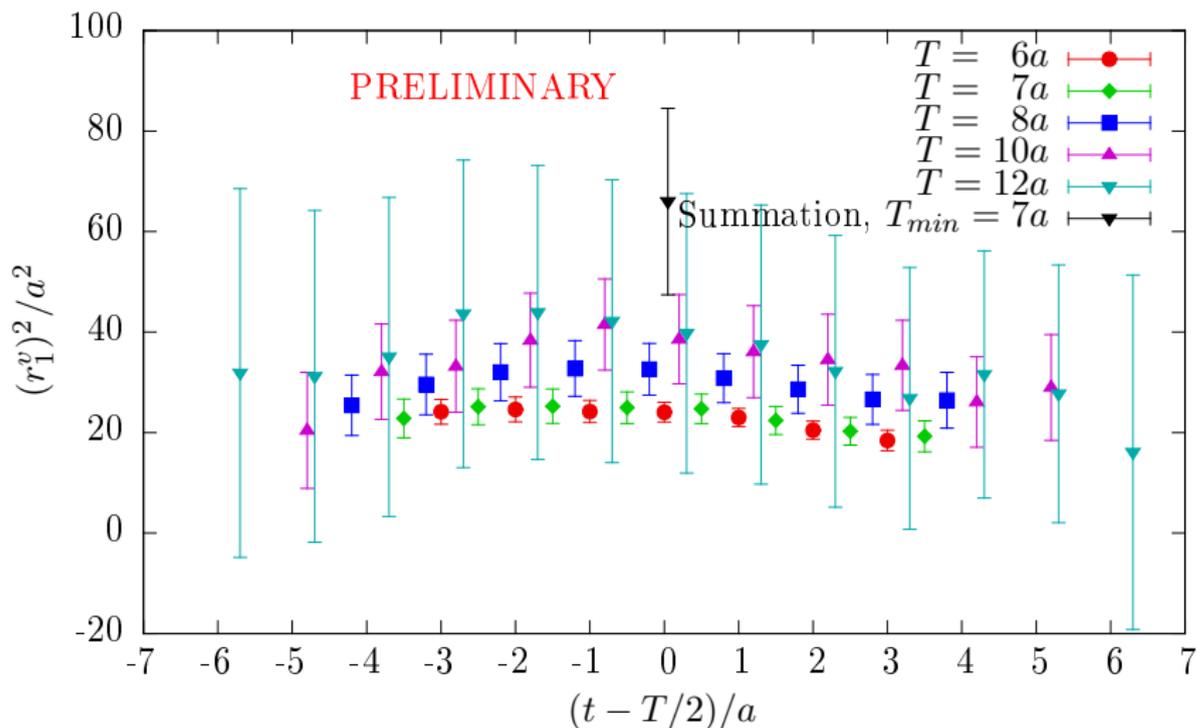
Ensemble

- BMW $N_f = 2 + 1$ 2HEX-clover, $L_s = L_t = 48$.
- $\beta = 3.31$, $a^{-1} = 1.697$ GeV, $a = 0.116$ fm .
- Physical pion mass, $m_\pi L = 4$.
- Five source-sink separations $T/a \in \{6, 7, 8, 10, 12\}$, which corresponds to T between 0.7 and 1.4 fm.
- 212 gauge configurations analysed
- AMA, using 96 sources per configuration with approximate propagators and one source for the bias correction.

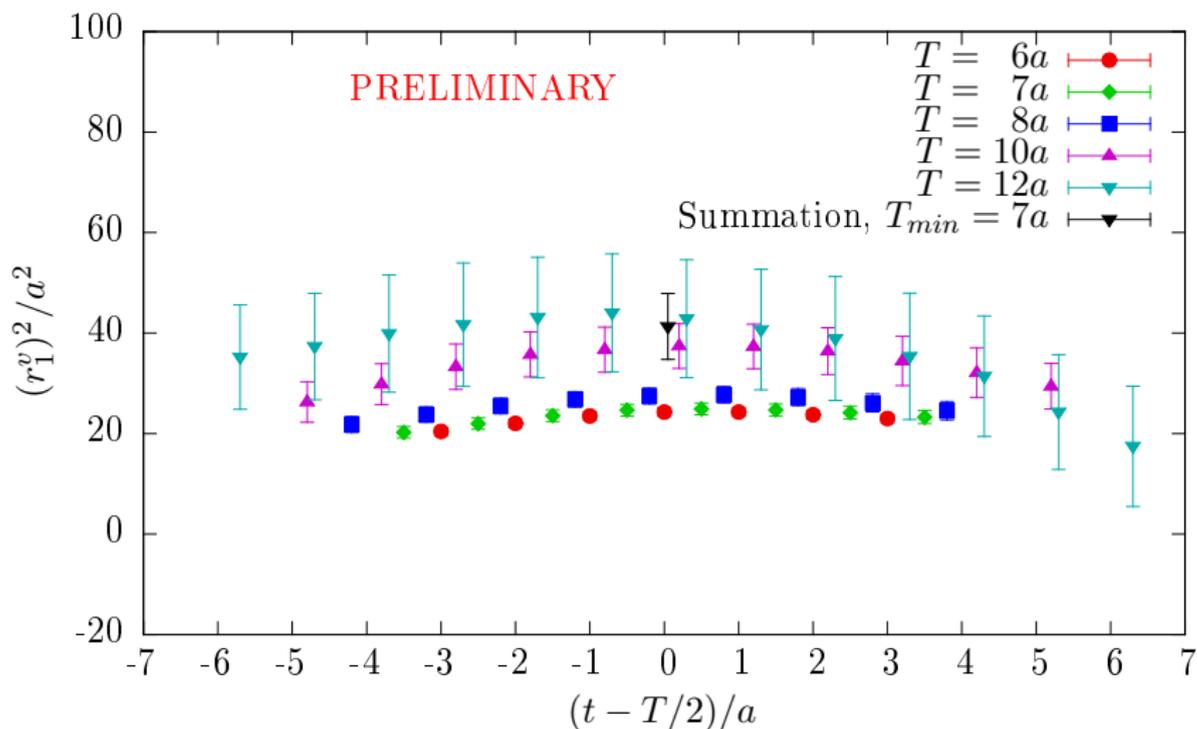
Isovector anomalous magnetic moment

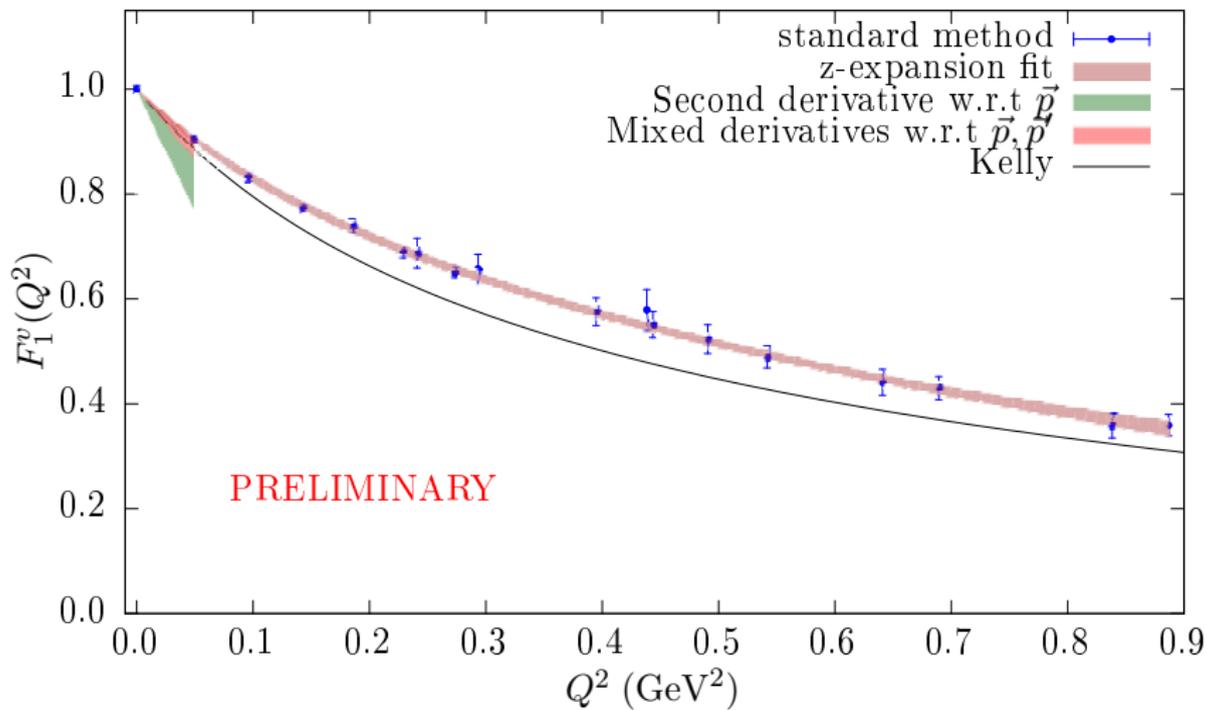


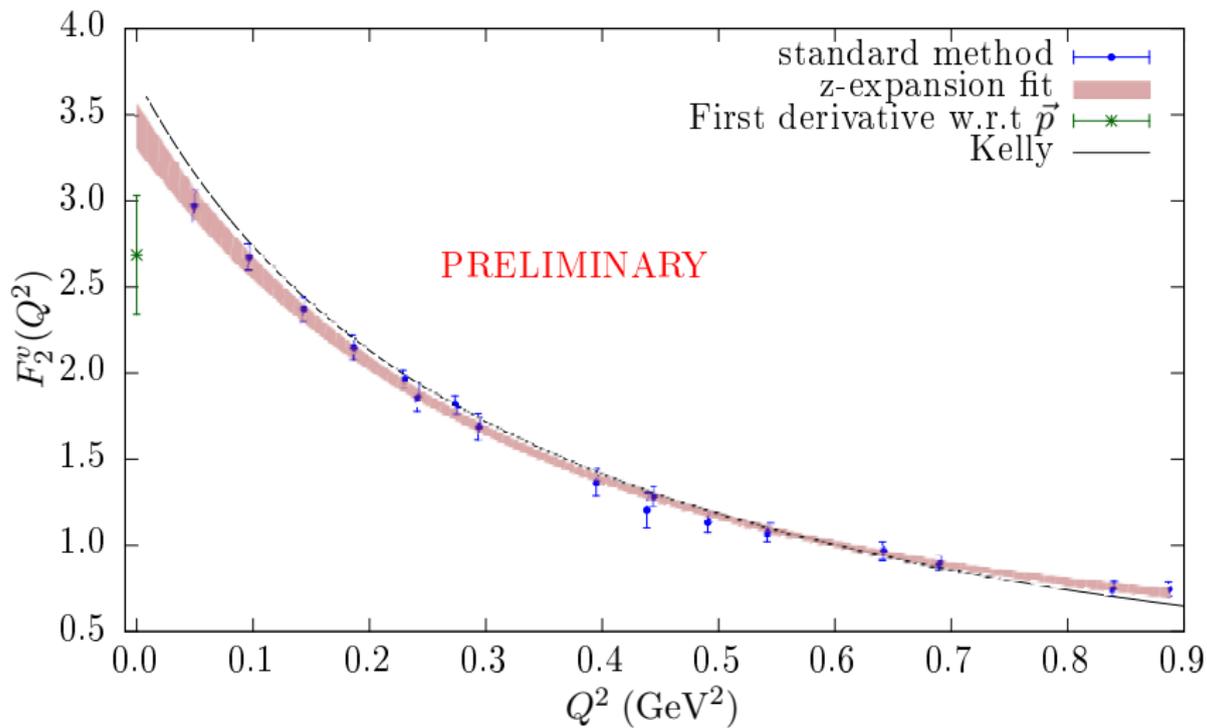
The Dirac radius (second momentum derivative w.r.t \vec{p})



The Dirac radius (mixed momentum derivatives w.r.t \vec{p} , \vec{p}')



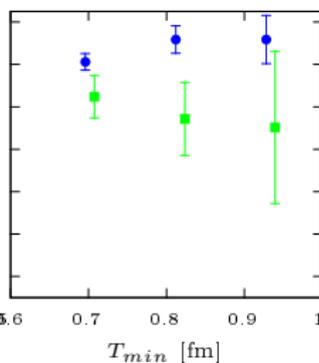
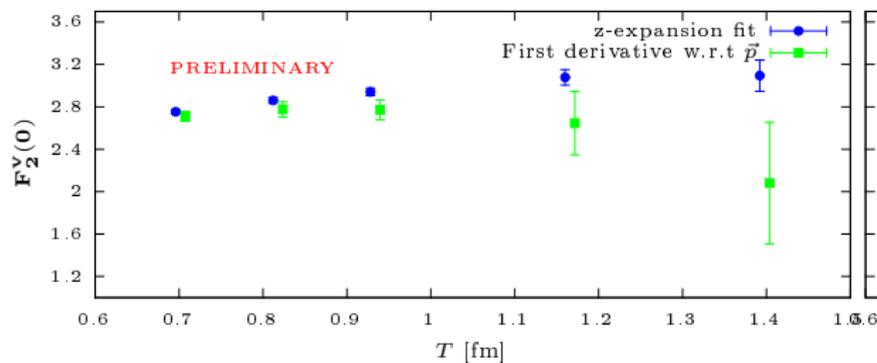
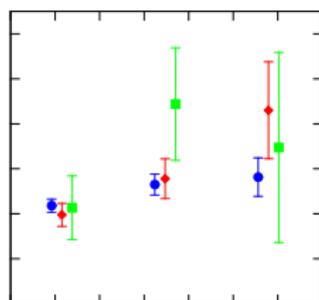
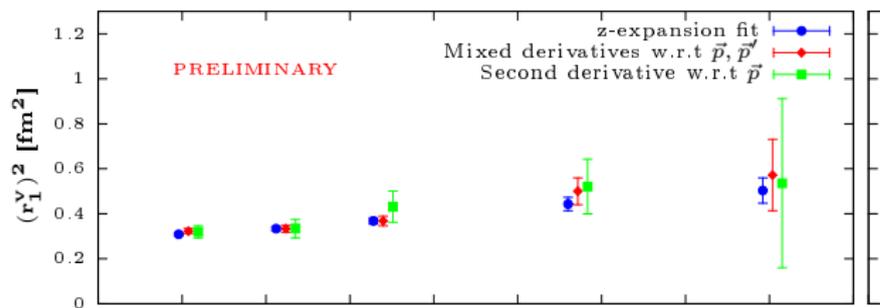




Summary plot

Plateau

Summation



Summary

- Our approach is based on the Rome method and is model independent.
- Using the mixed derivative approach, we were able to reduce the statistical uncertainty on $[r_1^v]^2$ and obtain results consistent with fitting to form factors.
- We also extended this method to the axial radius and form factor at zero Q^2 .
- This method may help lead to reliable lattice calculations of the proton radius and to better understanding of the radius puzzle.