

Kramers-Wannier duality with Abelian Color Fluxes for the $SU(2)$ principal chiral

`christof.gattringer@uni-graz.at`

The idea of dual variables

- In several theories one can solve the complex action problem at finite μ by exactly mapping the partition sum to dual variables
- Dual variables are generalized world-lines for matter fields and generalized world-sheets for gauge degrees of freedom
- Often several different dual representations are possible¹
- Symmetries of the underlying theory are converted into constraints for the dual variables
- Conserved charges become manifest as winding numbers of the dual variables
- Non-abelian d.o.f.s are a challenge which we here address with the new concept of Abelian Color Fluxes (ACF) for the example of the SU(2) principal chiral model²
 - How are non-abelian symmetries manifest in the dual representation?
 - How are charges represented?
 - What is the structure after a full Kramers-Wannier duality transformation?

¹ For a different dualization of the SU(2) PCR see: Rindlisbacher & de Forcrand, PoS LATTICE2015

² ACF can be generalized to gauge theories: NPB 913 (2017) and talk by Carlotta Marchis

SU(2) principal chiral model

- Partition sum:

$$Z = \int D[U] e^{-S} \quad , \quad \int D[U] = \prod_x \int_{\text{SU}(2)} dU_x$$

- Action with two charges coupled to chemical potentials μ_1 and μ_2

$$S = -\frac{J}{2} \sum_{x,\nu} \text{Tr} \left[e^{\delta_{\nu,d} \sigma^3 [\mu_1 + \mu_2]/2} U_x e^{\delta_{\nu,d} \sigma^3 [\mu_1 - \mu_2]/2} U_{x+\hat{\nu}}^\dagger \right. \\ \left. + e^{-\delta_{\nu,d} \sigma^3 [\mu_1 - \mu_2]/2} U_x^\dagger e^{-\delta_{\nu,d} \sigma^3 [\mu_1 + \mu_2]/2} U_{x+\hat{\nu}} \right]$$

- Using $U_x^{11} = U_x^{22*}$ and $U_x^{12} = -U_x^{21*}$ we find

$$S = -J \sum_{x,\nu} \sum_{a,b=1}^2 M_\nu^{ab} U_x^{ab} U_{x+\hat{\nu}}^{ab*}$$

with

$$M_\nu^{11} = e^{\delta_{\nu,d} \mu_1} \quad , \quad M_\nu^{12} = e^{\delta_{\nu,d} \mu_2} \quad , \quad M_\nu^{21} = e^{-\delta_{\nu,d} \mu_2} \quad , \quad M_\nu^{22} = e^{-\delta_{\nu,d} \mu_1}$$

Abelian Color Fluxes (ACF)

- We expand Z in $U_x^{ab} U_{x+\hat{\nu}}^{ab\star}$, referred to as "Abelian Color Fluxes"

$$\begin{aligned}
 Z &= \int D[U] e^{J \sum_{x,\nu} \sum_{a,b} M_\nu^{ab} U_x^{ab} U_{x+\hat{\nu}}^{ab\star}} = \int D[U] \prod_{x,\nu} \prod_{a,b} e^{J M_\nu^{ab} U_x^{ab} U_{x+\hat{\nu}}^{ab\star}} \\
 &= \int D[U] \prod_{x,\nu} \prod_{a,b} \sum_{l_{x,\nu}^{ab}=0}^{\infty} \frac{(J M_\nu^{ab})^{l_{x,\nu}^{ab}}}{l_{x,\nu}^{ab}!} \left(U_x^{ab} U_{x+\hat{\nu}}^{ab\star} \right)^{l_{x,\nu}^{ab}} \\
 &= \sum_{\{l\}} \prod_{x,\nu} \prod_{a,b} \frac{(J M_\nu^{ab})^{l_{x,\nu}^{ab}}}{l_{x,\nu}^{ab}!} \prod_x \int_{\text{SU}(2)} dU_x \prod_{a,b} \left(U_x^{ab} \right)^{\sum_\nu l_{x,\nu}^{ab}} \left(U_x^{ab\star} \right)^{\sum_\nu l_{x-\hat{\nu},\nu}^{ab}}
 \end{aligned}$$

- The $\text{SU}(2)$ matrices are integrated out with some explicit representation

$$U_x = \begin{pmatrix} \cos \theta_x e^{i\alpha_x} & \sin \theta_x e^{i\beta_x} \\ -\sin \theta_x e^{-i\beta_x} & \cos \theta_x e^{-i\alpha_x} \end{pmatrix}, \quad \alpha_x, \beta_x \in [-\pi, \pi], \quad \theta_x \in [0, \pi/2]$$

- Constraints and additional weights

$$\int d\alpha_x, \int d\beta_x \Rightarrow 2 \text{ constraints per site}, \quad \int d\theta_x \Rightarrow \text{site weights } W_H$$

Worldline representation

- Partition sum

$$Z = \sum_{\{l\}} e^{\mu_1 \sum_x [l_{x,d}^{11} - l_{x,d}^{22}]} e^{\mu_2 \sum_x [l_{x,d}^{12} - l_{x,d}^{21}]} W_B[l] W_H[l]$$

$$\prod_x \delta \left(\sum_{\nu} [(l_{x,\nu}^{11} - l_{x,\nu}^{22}) - (l_{x-\hat{\nu},\nu}^{11} - l_{x-\hat{\nu},\nu}^{22})] \right) \delta \left(\sum_{\nu} [(l_{x,\nu}^{12} - l_{x,\nu}^{21}) - (l_{x-\hat{\nu},\nu}^{12} - l_{x-\hat{\nu},\nu}^{21})] \right)$$

- Introduce new dual variables $k_{x,\nu}^1, k_{x,\nu}^2 \in \mathbb{Z}$ and $h_{x,\nu}^1, h_{x,\nu}^2 \in \mathbb{N}_0$

$$l_{x,\nu}^{11} - l_{x,\nu}^{22} = k_{x,\nu}^1$$

$$j_{x,\nu}^{12} - j_{x,\nu}^{21} = k_{x,\nu}^2$$

$$l_{x,\nu}^{11} + l_{x,\nu}^{22} = |k_{x,\nu}^1| + 2h_{x,\nu}^1$$

$$j_{x,\nu}^{12} + j_{x,\nu}^{21} = |k_{x,\nu}^2| + 2h_{x,\nu}^2$$

- Final worldline form

$$Z = \sum_{\{k,h\}} e^{\mu_1 \sum_x k_{x,d}^1} e^{\mu_2 \sum_x k_{x,d}^2} W_B[k, h] W_H[k, h] \prod_x \delta(\nabla k_x^1) \delta(\nabla k_x^2)$$

$$\nabla k_x^\alpha \equiv \sum_{\nu} [k_{x,\nu}^\alpha - k_{x-\hat{\nu},\nu}^\alpha]$$

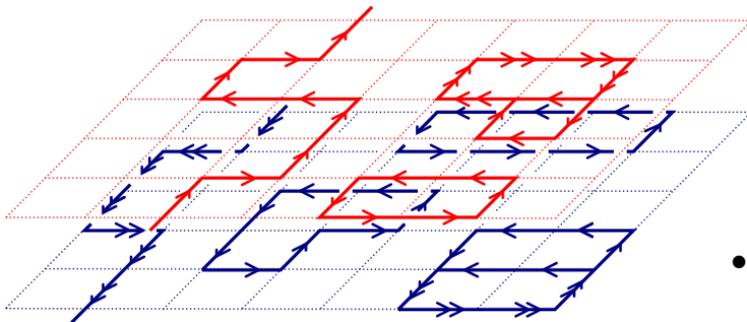
Geometrical interpretation

- Final form of worldline partition sum

$$Z = \sum_{\{k,h\}} e^{\mu_1 N_T \omega[k^1]} e^{\mu_2 N_T \omega[k^2]} W_B[k, h] W_H[k, h] \prod_x \delta(\nabla k_x^1) \delta(\nabla k_x^2)$$

$$W_B[k, h] = \prod_{x,\nu} \prod_{\alpha=1}^2 \frac{J^{D_{x,\nu}^\alpha}}{(D_{x,\nu}^\alpha - h_{x,\nu}^\alpha)! h_{x,\nu}^\alpha!} \quad \omega[k^\alpha] = \text{temporal winding}$$

$$W_H[k, h] = \prod_x \frac{\prod_\alpha \left(\frac{1}{2} \sum_\nu [D_{x,\nu}^\alpha - D_{x-\hat{\nu},\nu}^\alpha] \right)!}{\left(1 + \sum_\alpha \sum_\nu [D_{x,\nu}^\alpha - D_{x-\hat{\nu},\nu}^\alpha] \right)!} \quad D_{x,\nu}^\alpha \equiv |k_{x,\nu}^\alpha| + 2h_{x,\nu}^\alpha$$

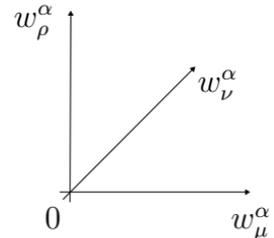
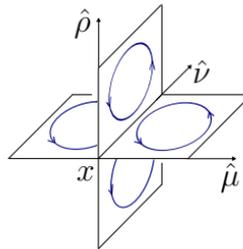


- Theory of 2 conserved fluxes k^1, k^2
- Weight W_H and auxiliary variables h^1, h^2 implement $SU(2)$

Steps for a full dualization (Kramers - Wannier)

- Resolve the constraints by generating the conserved fluxes $k_{x,\nu}^\alpha$ with new, unconstrained variables:
 - Fluxes $n_{x,\mu\nu}^\alpha \in \mathbb{Z}$ around plaquettes
 - Global defect fluxes w_ν^α for the winding

$$k_{x,\nu}^\alpha = \sum_{\rho: \nu < \rho} [n_{x,\nu\rho}^\alpha - n_{x-\hat{\nu},\nu\rho}^\alpha] - \sum_{\mu: \mu < \nu} [n_{x,\mu\nu}^\alpha - n_{x-\hat{\mu},\mu\nu}^\alpha] + \theta(x,\nu) w_\nu^\alpha$$



- Identify counterparts on the dual lattice for the new variables $n_{x,\mu\nu}^\alpha$ and w_ν^α , as well as for the auxiliary variables $h_{x,\mu\nu}^\alpha$.

Dualization in 3 dimensions (other dimensions equivalent)

- Variables on the dual lattice

$$n_{x,\mu\nu}^\alpha \longrightarrow l_{x,\mu}^\alpha \in \mathbb{Z} \quad \text{dynamical variables on dual links}$$

$$h_{x,\mu}^\alpha \longrightarrow a_{x,\mu\nu}^\alpha \in \mathbb{N}_0 \quad \text{auxiliary variables on dual plaquettes}$$

$$w_\mu^\alpha \longrightarrow b_{\mu\nu}^\alpha \in \mathbb{Z} \quad \text{boundary condition on dual plaquettes}$$

$$P_{x,\mu\nu}^\alpha \equiv |l_{x,\mu}^\alpha - l_{x+\hat{\mu},\nu}^\alpha - l_{x+\hat{\nu},\mu}^\alpha + l_{x,\nu}^\alpha + \theta(x, \mu, \nu) b_{\mu\nu}^\alpha| + 2 a_{x,\mu\nu}^\alpha$$

- Fully dualized partition sum in 3D

$$Z = \sum_{\{b\}} e^{\mu_1 N_T b_{12}^1} e^{\mu_2 N_T b_{12}^2} \sum_{\{l,a\}} M_B[l, a, b] M_H[l, a, b]$$

$$M_B[l, a, b] = \prod_{\text{plaquettes } p} \prod_{\alpha=1}^2 \frac{J P_p^\alpha}{(P_p^\alpha - a_p^\alpha)! a_p^\alpha!}$$

$$M_H[l, a, b] = \prod_{\text{cubes } \mathcal{C}} \frac{\prod_{\alpha} \left(\frac{1}{2} \sum_{p \in \partial \mathcal{C}} P_p^\alpha \right)!}{\left(1 + \frac{1}{2} \sum_{\alpha} \sum_{p \in \partial \mathcal{C}} P_p^\alpha \right)!}$$

strong coupling \Leftrightarrow weak coupling

Summary

- We explore new representations of the $SU(2)$ principal chiral model with chemical potentials coupled to two conserved charges
- Worldline representation
 - We expand the Boltzmann factor in Abelian Color Fluxes
 - All terms of the partition sum are real and positive: complex action problem solved
 - Theory of two worldlines interacting with each other via site weights $\Rightarrow SU(2)$
- Full Kramers - Wannier dualization
 - Constraints can be resolved by introducing new plaquettes variables and defect lines
 - Dual theory of links variables, with interactions on plaquettes and on cubes for implementing $SU(2)$
 - Dualization in the full sense: strong coupling \Leftrightarrow weak coupling
- Exploratory study of simulating the different forms has started