Four-Fermi Theories in 3 Dimensions: Critical behaviour

Andreas Wipf

Theoretisch-Physikalisches Institut
Friedrich-Schiller-University Jena

seit 1558

Four-Fermi field theories

- continuum models:
  - renormalizable in large-N_f expansion Rosenstein, Warr, Park; Gawedzki, Kupiainen;
  - non-Gaussian UV fixed point(s) (cp. QG) da Veiga, de Calan, Magnen, Seneor
  - large N_f expansion Gracey; Hands, Kocic, Kogut; Dateki . . .
  - recent FRG-results on critical behaviour Gies, Braun, Janssen, Herbut, Borchard, Knorr,. . .
- lattice theories:
  - critical exponents/flavour number (staggered) . . . Kärkkäinen et al.; Hands et al.. . .
  - fermion bag model (staggered) Chandrasekharan, Li
  - domain wall fermions talk by S. Hands
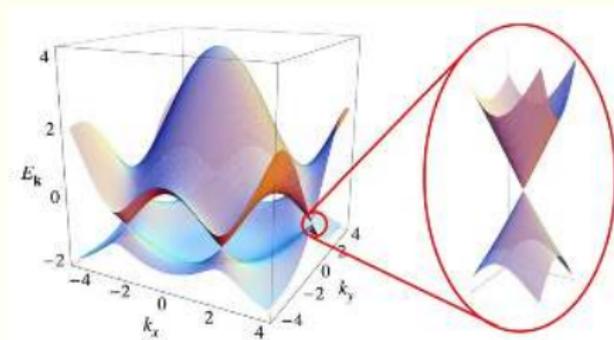 - quantum Monte Carlo Hesselmann et al; Li; Wang; Otsuka, Yunoki, Sorella

condensed matter systems

- low energy description of tight binding
- honeycomb lattice:
  two atoms per unit cell, two Dirac points
  $\Rightarrow$ 4-component (reducible) spinor field
- interaction-driven metal-insulator transition $\rightarrow$ Gross-Neveu (GN) model

electronic dispersion in honeycomb lattice (from Castro Neto et al.)

4-component Dirac fermions (previous simulations)

- from dimensional reduction $d = 4 \rightarrow d = 3$
- reducible 4-dimensional gamma-matrices $\Gamma_\mu, \ \mu = 1, 2, 3$
- notion of chiralities

$$\Gamma_4, \Gamma_5, \Gamma_{45} = i\Gamma_4\Gamma_5, \quad \{\Gamma_\mu, \Gamma_4\} = \{\Gamma_\mu, \Gamma_5\} = [\Gamma_\mu, \Gamma_{45}] = 0$$

- N_f reducible 4-Dirac spinors: $\Psi = (\psi_1, \ldots, \psi_{N_f})^T$

$$i\bar{\Psi}\slashed{\partial}\Psi = i\bar{\Psi}\left(\Gamma^\mu\partial_\mu \otimes \mathbb{1}_{N_f}\right)\Psi$$

- $U(2N_f)$ symmetry generated by $\{\mathbb{1}, \Gamma_{45}, i\Gamma_4, i\Gamma_5\} \otimes A, \quad A^\dagger = -A$

- reducible Thirring-model

$$\mathcal{L}_{\mathrm{Th}} = \bar{\Psi}\mathrm{i}\slashed{\partial}\Psi - \frac{g^2}{2N_{\mathrm{f}}}(\bar{\Psi}\Gamma^\mu\Psi)^2, \quad \mathbb{Z}_2 \times SU(2N_{\mathrm{f}}) \text{ symmetry}$$

- $\langle\bar{\Psi}\Psi\rangle$ order parameter for continuous symmetry (not $\langle\bar{\Psi}\Gamma_{45}\Psi\rangle$)
- Gross-Neveu model

$$\mathcal{L}_{\mathrm{GN}} = \bar{\Psi}\mathrm{i}\slashed{\partial}\Psi + \frac{g^2}{2N_{\mathrm{f}}}(\bar{\Psi}\Psi)^2, \quad \mathbb{Z}_2 \otimes U(N_{\mathrm{f}}) \times U(N_{\mathrm{f}}) \text{ symmetry}$$

- $\langle\bar{\Psi}\Psi\rangle$ order parameter for discrete chiral $\mathbb{Z}_2$

2-component (irreducible) Dirac spinors

- irreducible γ^μ are 2×2 matrices, no chirality
- N_f^{ir} flavours of 2-component Dirac spinors, $\Psi = (\psi_1, \ldots, \psi_{N_f^{ir}})$
- $\mathbb{Z}_2$-parity and flavour-$U(N_f^{ir})$ invariant theories

$$\mathcal{L}_{GN} = \bar\Psi i \partial\!\!\!/ \Psi + \frac{g^2}{2N_f}(\bar\Psi\Psi)^2 \quad , \quad \mathcal{L}_{Th} = \bar\Psi i \partial\!\!\!/ \Psi - \frac{g^2}{2N_f}(\bar\Psi\gamma^\mu\Psi)^2$$

- $\langle\bar\Psi\Psi\rangle$ order parameter for parity (chiral Ising class)

$$x_3 \to -x_3 \Longrightarrow \psi_a \to i\gamma_3\psi_a, \ \ \bar\psi_a \to i\bar\psi_a\gamma_3 \Longrightarrow \langle\bar\Psi\Psi\rangle \to -\langle\bar\Psi\Psi\rangle$$

- reducible vs. irreducible case: consider $N_f = 1$ or $N_f^{ir} = 2$

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}, \; \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Longrightarrow \bar\psi \Gamma^\mu \partial_\mu \psi = \bar\psi_1 \gamma^\mu \partial_\mu \psi_1 - \bar\psi_2 \gamma^\mu \partial_\mu \psi_2$$

- to standard kinetic term $\Longrightarrow \bar\Psi\Psi|_{\mathrm{red}} = \Psi \Gamma_{45} \Psi|_{\mathrm{irred}}$ breaks $U(N_f^{ir})$
- $m\bar\Psi\Psi|\mathrm{red}$ alternating sign of m
  $\Rightarrow$ expect no parity breaking in reducible case
- Hubbard-Stratonovich transformation

$$\mathcal{L}_{\mathrm{GN}} = \bar\Psi\big(\mathrm{i}D \otimes \mathbb{1}_{N_f}\big)\Psi + \frac{N_f}{2g^2}\sigma^2, \qquad D = \slashed\partial - \sigma$$

$$\mathcal{L}_{\mathrm{Th}} = \bar\Psi\big(\mathrm{i}\slashed{D} \otimes \mathbb{1}_{N_f}\big)\Psi + \frac{N_f}{2g^2} V_\mu V^\mu, \quad \slashed{D} = \slashed\partial - \mathrm{i}\slashed{V}$$

- no sign problem for reducible models:

$$\det\left(\Gamma^\mu(i\partial_\mu + V_\mu)\right) = \left(\det\gamma^\mu(i\partial_\mu + V_\mu)\right)^2 \geq 0$$
$$\det(\Gamma^\mu\partial_\mu - \sigma) = |\det(\gamma^\mu\partial_\mu - \sigma)|^2 \geq 0$$

- reducible N_f Thirring-model $\Leftrightarrow$ irreducible $2N_f$ model
- reducible N_f Gross-Neveu-model $\equiv$ phase quenched irred. $2N_f$ model
- but: sign problem for
    - irreducible GN for all N_f^{ir}
    - irreducible Th with odd N_f^{ir}
- $N_f^{ir} = 1$: Thirring model $\equiv$ Gross-Neveu model

  dual formulation without sign problem
  talks by Wellegehausen

- small N_f limit: $\langle\bar{\psi}\psi\rangle \neq 0$ for $N_f^{ir} = 1$
- large-N_f limit:

$$\text{Thirring:} \quad \langle\bar{\Psi}\Psi\rangle_{m\to 0} = 0$$

$$\text{Gross-Neveu:} \quad \langle\bar{\Psi}\Psi\rangle_{m\to 0} = \pm\frac{2\pi}{g_{\text{ren}}^2} \quad \text{for} \quad g^2 > g_{\text{crit}}^2$$

$\Rightarrow$ PT for all GN, critical flavour number for Thirring   talks by B. Wellegehausen and Hands

- Fierz-rearrangement

$$(\bar{\Psi}\gamma^\mu\Psi)(\bar{\Psi}\gamma_\mu\Psi) = -2\sum(\bar{\psi}^a\psi^b)(\bar{\psi}^b\psi^a) - (\bar{\Psi}\Psi)^2$$

$$\Rightarrow \mathcal{L}_{\text{Th}} = \bar{\Psi}(i\slashed{\partial} + iT + i\phi)\Psi + \frac{N_f^{ir}}{4g^2}\operatorname{tr}T^2 + \frac{N_f^{ir}}{2g^2}\phi^2$$

- N_f^{ir}-dimensional hermitian matrix field, ϕ real $\Rightarrow$ dualization
- severe sign problem

- problems with previous lattice studies
    - Wilson fermions break parity and $U(2N_f)$
    - staggered fermions: break $SU(2N_f)$, severe tast-breaking   Giedt et al.
- chiral SLAC fermions   Drell, Weinstein, Yankielowicz
- ∂_μ real hermitean, non-local
- identical internal symmetries as continuum-models
- successfully tested/applied
    - scalar field theories (lattice perturbation theory)   G. Bergner
    - non-linear ON)-models   step scaling function $\sigma = 1.2604(13)$ vs. $\sigma = 1.261210$
    - various susy Yukawa models   Ward indentities, masses, goldstino
    - used in pseudo-spectral methods
- fails for gauge theories!

- this talk: reducible models
- Simulation parameters
  - lattice size $V = L \times (L-1) \times (L-1)$, $\quad L = 8, 12, 16, 20, 24$
  - $N_f = 1, 2, 3, \ldots$
  - rHMC, N_f pseudo fermions
- order parameter
  - Schwinger-Dyson equation

$$\langle \bar{\Psi}\Psi \rangle = -\lambda_{GN}\langle \sigma \rangle, \quad \sigma = \frac{1}{V}\sum_x \sigma(x)$$
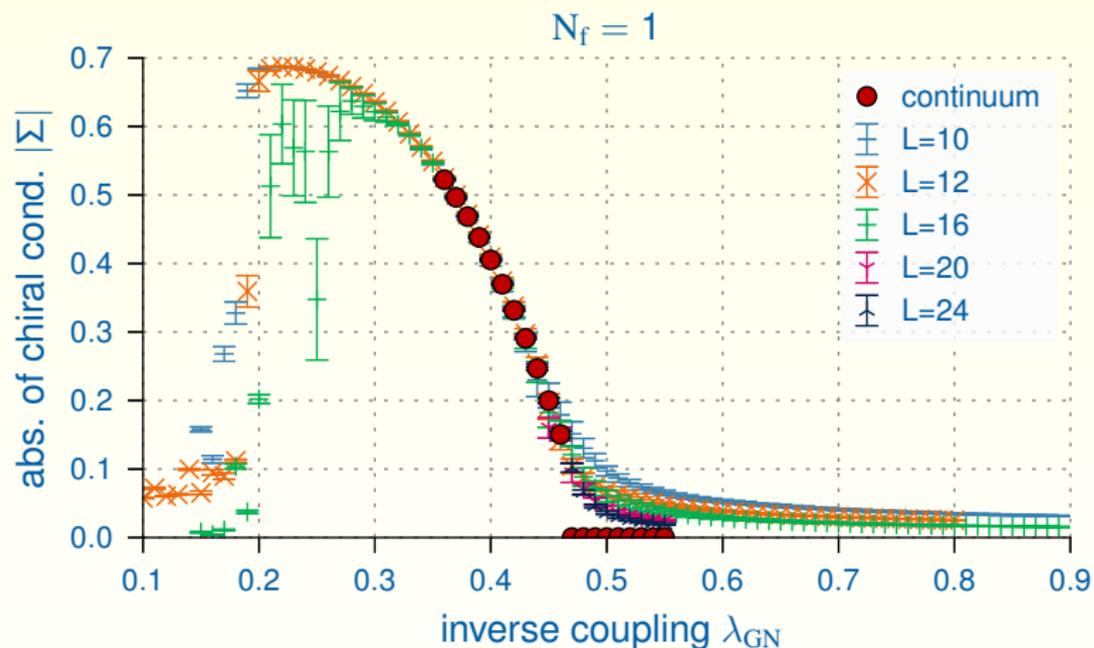
lattice artifact phase at strong coupling: filling factor ≈ 1 talk by B. Wellegehausen
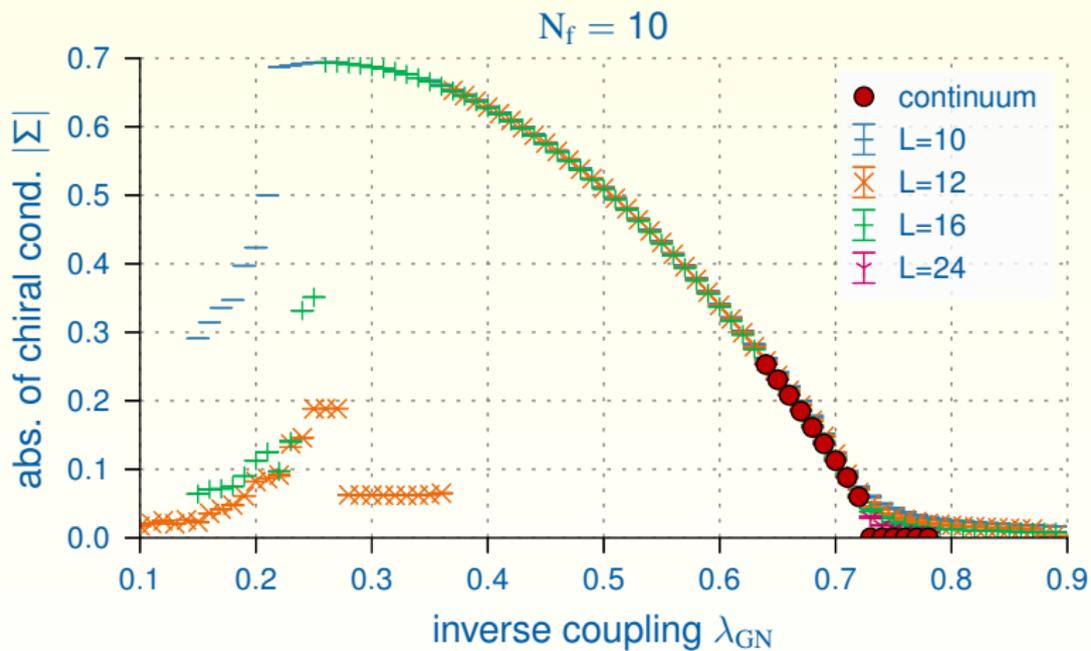
- critical exponents from finite size scaling (at criticality)
  - $\Rightarrow$ need precise value for $\lambda_{cr} = \lim_{L \to \infty} \lambda_{cr}(L)$
    - $\lambda_{cr}(L)$ from peak of $\chi(L)$, expect

$$\lambda_{cr}(L) = \lambda_{cr} + a' L^{-1/\nu}(1 + b' L^{-\omega})$$
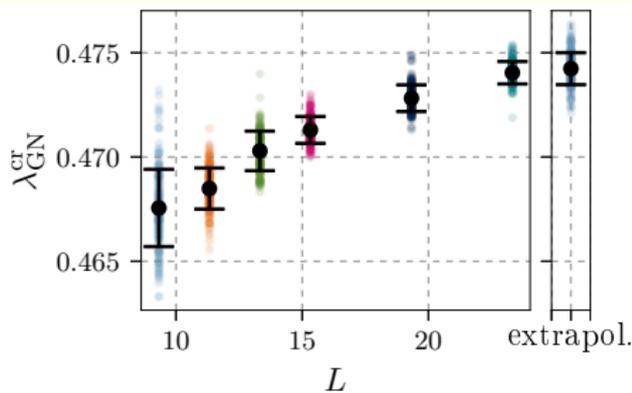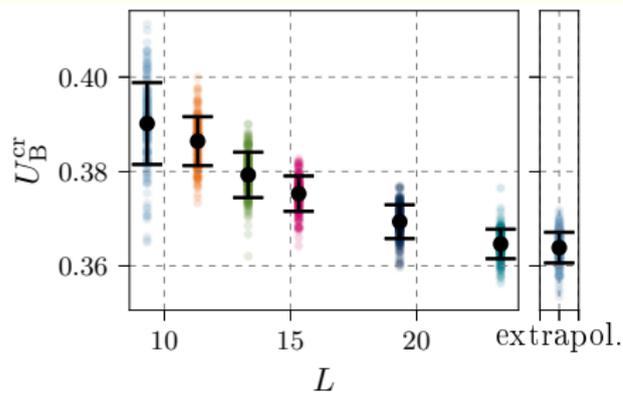
    - $\lambda_{cr}(L)$ from Binder cummulant

$$U_B = 1 - \frac{\langle \sigma^4 \rangle}{3\langle \sigma^2 \rangle^2}$$

    - intersection point on lattice L and lattice bL

$$U_B(L, b) = U_B \left(1 + c\, L^{-\omega} \frac{1 - b^{-\omega - 1/\nu}}{1 - b^{-1/\nu}} \right)$$
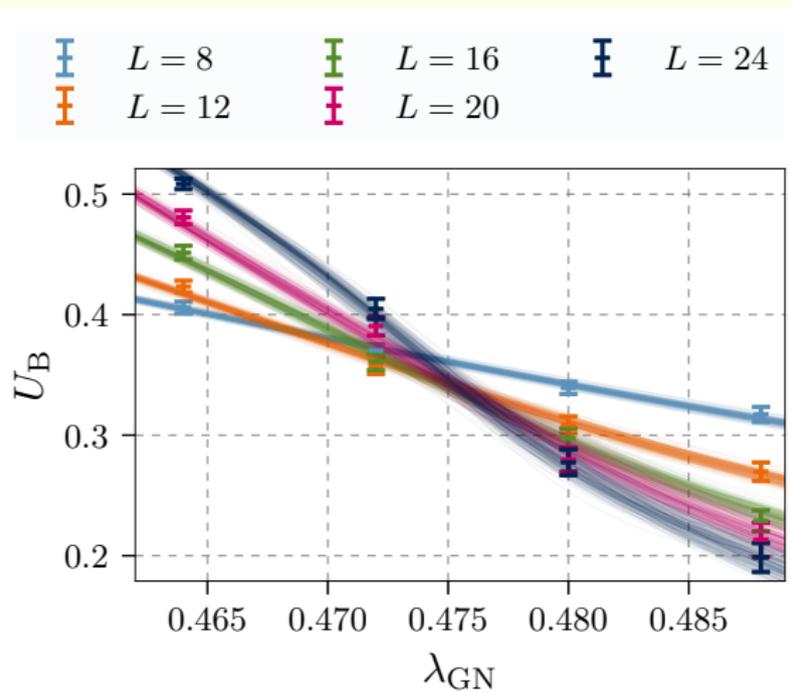
Binder cummulants for $N_f = 1$

N_f	λ_{cr} Binder	λ_{cr} σ-fit	U_B^{cr}
1	0.4742(7)	0.475(1)	0.364(3)
2	1.215(1)	1.216(4)	0.333(3)

- order parameter
$$\Sigma = \langle |\sigma| \rangle \propto L^{-\beta/\nu}$$

- susceptibility
$$\chi = V \left(\langle \sigma^2 \rangle - \langle |\sigma| \rangle^2 \right) \propto L^{\gamma/\nu}$$

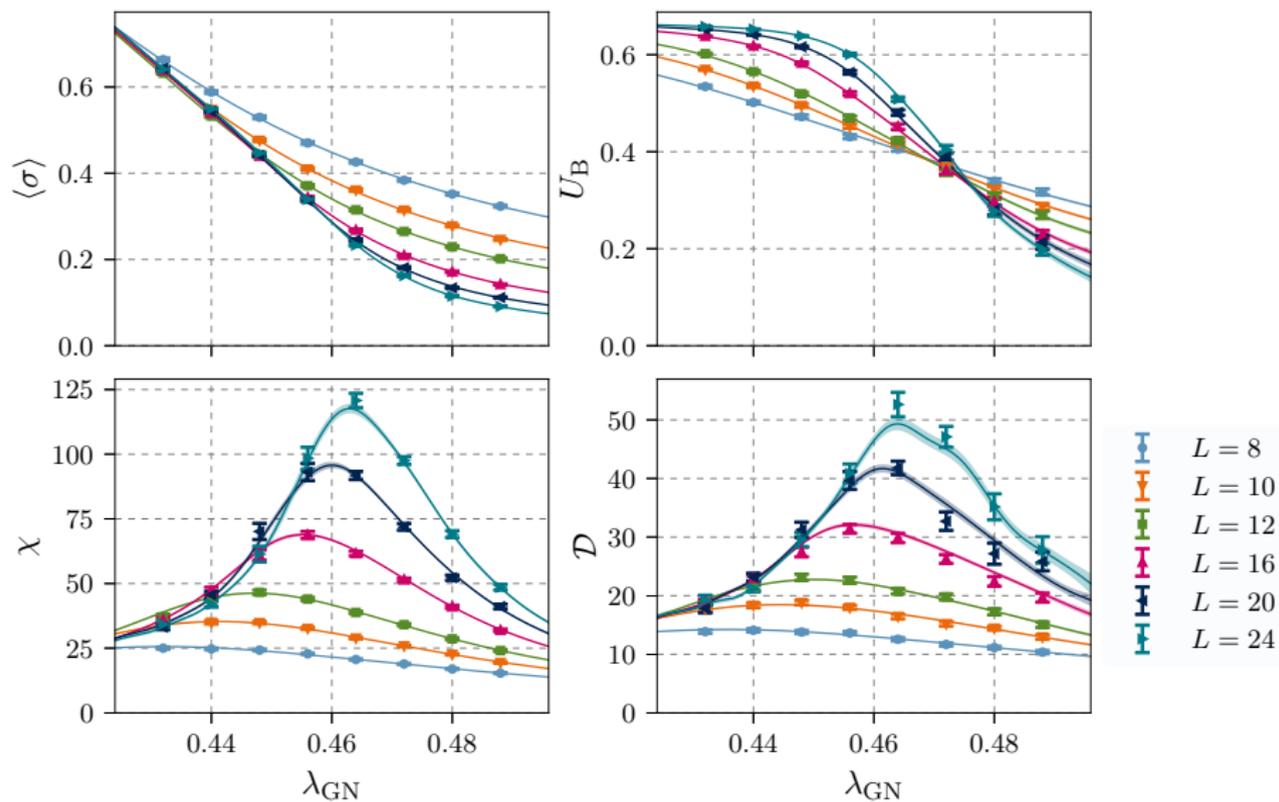- ν from slopes of U_B or $\log\langle |\sigma| \rangle$

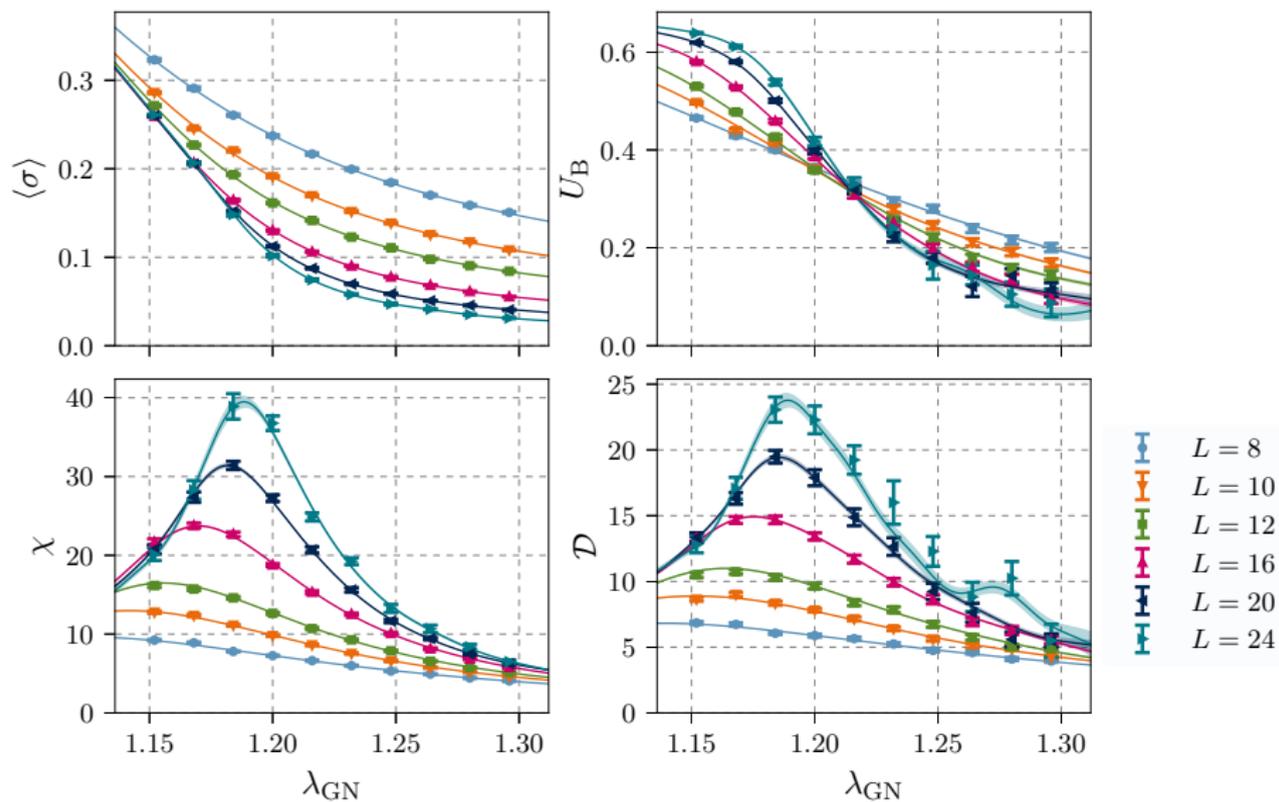$$\frac{\partial U_{\mathrm{B}}}{\partial \lambda} = a L^{-\nu} \left(1 + b L^{-\omega} \right)$$
$$\mathcal{D} = \frac{\partial \log\langle |\sigma| \rangle}{\partial \lambda} = \frac{\langle |\sigma| S_{\mathrm{bos}} \rangle}{\langle |\sigma| \rangle} - \langle S_{\mathrm{bos}} \rangle \propto L^{1/\nu}$$

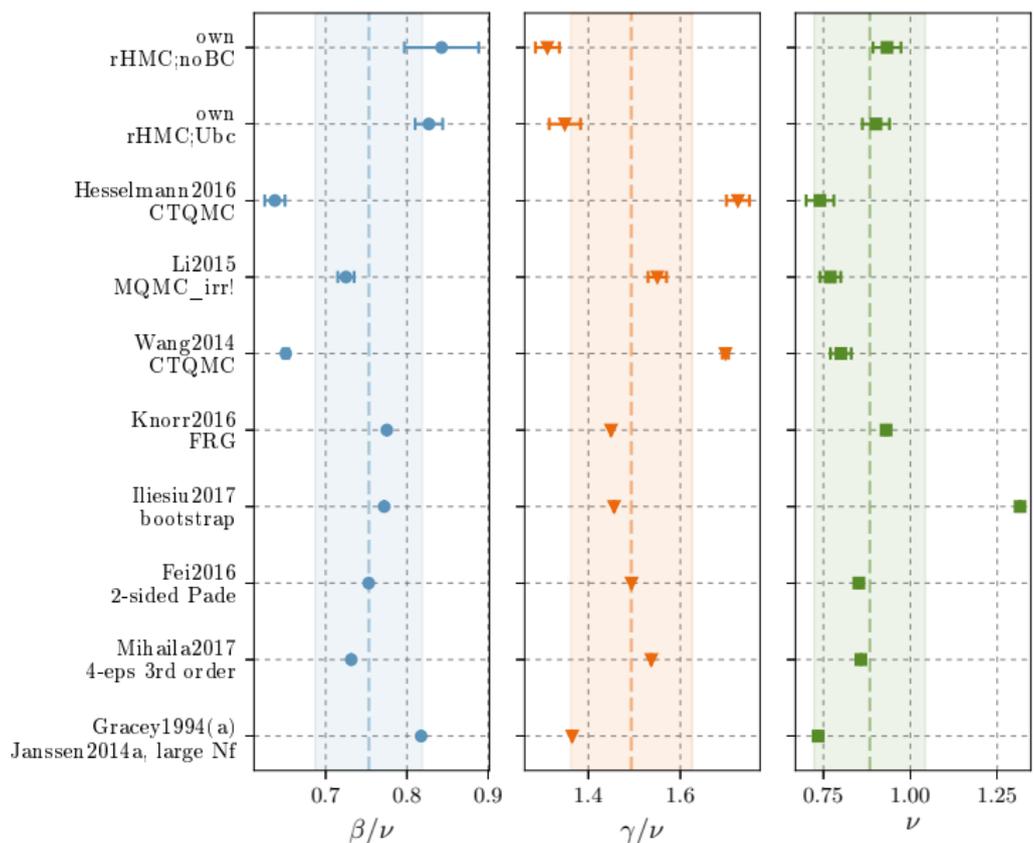- compare with phenomenological renormalization   Engels, Fingberg, Miller
- hyperscaling relation: satisfied, up to approx. ± 0.04
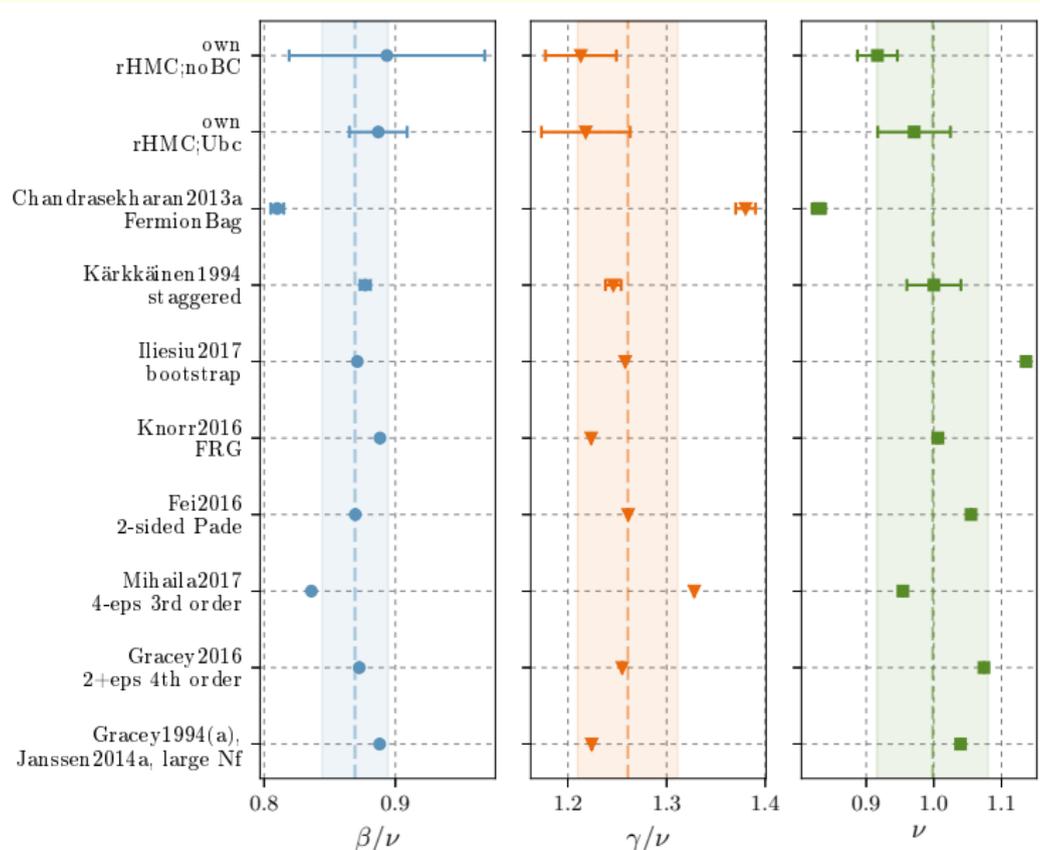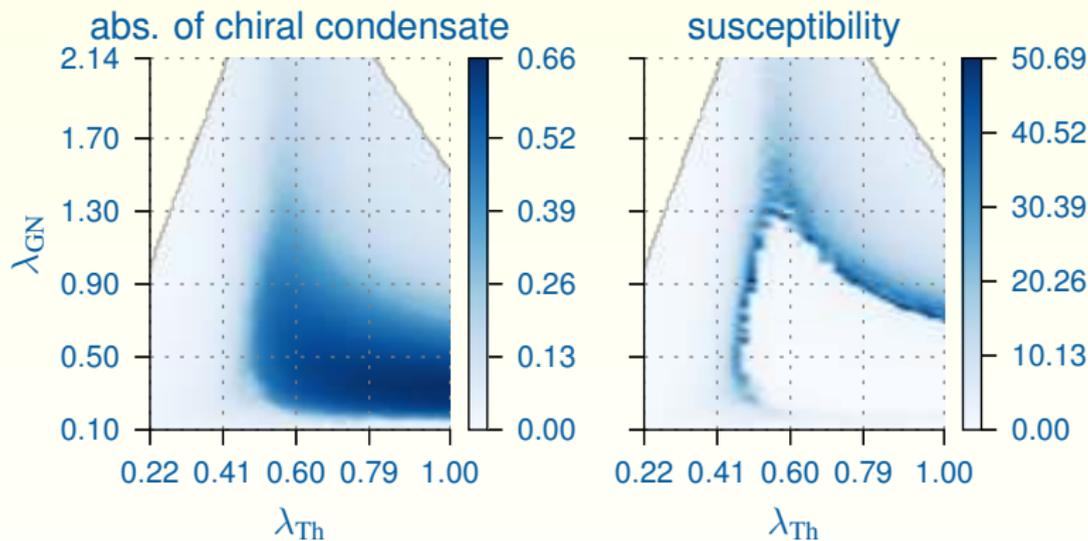
$$2\frac{\beta}{\nu} + \frac{\gamma}{\nu} - 3 = 0$$

- order parameter and susceptibility on coarse lattice

Thirring model: is there no phase transition even for $N_f = 1$?

- in this (and following) talk
  - successful simulations of four-Fermi theories with chiral fermions
  - irreducible models may have sign problem
    under analytical/numerical control (dual formulation)
    understanding of lattice artifact phase
  - exponents of GN models from finite size scaling
  - large N_f critical exponents $\neq$ MF Ising model:

$$U_{\text{eff}} = U_0 - h\sigma + ta_2\sigma^2 + a_3|\sigma|^3 \quad \mathbb{Z}_2\text{symmetric, non-analytic}$$

  - critical N_f different for reducible and irreducible Thirring models
- possible applications/outlook
  - irreducible models with sign problem (dual variables)
  - interacting-Fermion systems in condensed matter (chiral Heisenberg, XY)
  - parity-restoring second order transition at finite T? Pisarski and Wilczek; Kocic and Kogut